$$\rho^{(2)} = A[H(A^2+1)+A^2]^{-2}\{k_1[H(A^2+1)-A^2] + 2D_1A^2H^{-1}\}\{g(x)-\tilde{y}\} + HA[H(A^2+1)+A^2]^{-2} \times \{2HA^3+4HA+2HA^{-1}+3A^3+3A\}$$

$$v_1^{(2)} = -Ak_0(A^2+1)^{-1}[g(x)-\tilde{y}]$$

$$v_2^{(2)} = -k_0A^{-2}(2A^2+1)(A^2+1)^{-1}[H(A^2+1) + A^2][g(x)-\tilde{y}] + k_0(A^2+1)^{-1}g_1(x)$$

$$N^{(2)} = k_0A^{-1}[g(x)-\tilde{y}]$$

$$T_p^{(2)} = D_0AH^{-1}(A^2+1)^{-1}[g(x)-\tilde{y}]$$

where

$$g_1(x) = (A^2+1)A^{-1}\{A^{-2}[A^4-H^2(A^2+1)^2] + 2[H(A^2+1)+A^2]^2 - H(A^2+1)\}x - (2A)^{-1}(A^2+1)^2[H(A^2+1)+A^2] \times \{2D_1H^{-1}+k_1A^{-2}[H(A^2+1)-A^2]\}x^2$$

Discussion

1) Expressions for the fluid streamlines and dust particle paths[7] can be obtained in the same manner as for the small-disturbance theory.[6]

2) The shock wave is bent toward the wedge surface [quadratic term in $g_1(x)$] under the influence of the dust particles.

3) Of practical interest, the surface pressure p_s on the wedge is given to second order as

$$p_s = A^2(A^2+1)^{-1} + \epsilon\{2H + A^2(A^2+1)^{-1} + k_1[H(A^2+1)+A^2]x\}$$

showing that it increases linearly with distance from the vertex of the wedge.

Acknowledgment

This work was supported by the Natural Sciences and Engineering Research Council of Canada under Grant A4484.

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A "Similarity" Solution for Laminar Swirling Core Flows

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Introduction

THE present investigation develops a linearized "similarity" solution, based on the Oseen assumption of a small axial disturbance velocity, for the far downstream region of swirling laminar core flows embedded in a uniform parallel following stream. The solution includes the cases of "weak" swirl where the pressure perturbation is sufficiently small for the axial and azimuthal fields to be effectively uncoupled; "strong" swirl with significant pressure coupling of these fields; and the special case of "very strong" swirl in jets transmitting zero, or at least very small flow force.

The approach used to obtain the linearized solution is similar to that employed by Batchelor[1] in obtaining a similarity solution for strong trailing vortices far downstream of their point of generation.

The swirling core flows considered are of interest in connection with modeling such practical applications as the flow downstream of rotating bodies (e.g., spinning projectiles and turbomachinery). These flows will be turbulent, whether in a real application or in the laboratory; nevertheless, it appears reasonable to investigate the laminar flow, particularly from the viewpoint of understanding the role of the swirl-induced pressure coupling and its effect on the axial flow.

"Similarity" Solutions for Swirling Laminar Core Flows Far Downstream

We shall consider a single axisymmetric swirling core and use cylindrical polar coordinates (r,ϕ,z) with corresponding velocity components (u,v,w) to investigate viscous development of the core in a uniform outer stream of velocity W_0 and pressure p_∞. If we consider the steady motion of a viscous incompressible flow, and assume that $\partial/\partial z \ll \partial/\partial r$; $u \ll w$, and $|w-W_0| \ll W_0$; then the equations of motion can be approximated by[1]:

$$\frac{\partial}{\partial z}(rw) + \frac{\partial}{\partial r}(ru) = 0 \tag{1}$$

$$W_0\frac{\partial v}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{2}$$

$$W_0\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3}$$

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \tag{4}$$

where p is the disturbance pressure due to the core flow, ν the kinematic viscosity, and ρ the uniform density of the fluid.

Received June 23, 1982; revision received Nov. 23, 1982.

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The equations are to be solved subject to the boundary conditions

$$u=v=\frac{\partial w}{\partial r}=\frac{\partial p}{\partial r}=0, \quad w,p \text{ finite at } r=0$$

$$u,v\rightarrow 0, \quad w\rightarrow W_0, \quad p\rightarrow p_\infty \text{ as } r\rightarrow\infty \tag{5}$$

The momentum-integral equations (for derivation, see Morton[2]):

$$2\pi\int_0^\infty \rho\{W_0(w-W_0)-\tfrac{1}{2}v^2\}r\mathrm{d}r=F \tag{6}$$

$$2\pi\int_0^\infty \rho r^2 v W_0 \mathrm{d}r=G \tag{7}$$

complete the specification of the solution. The constant F represents the invariant "flow force," or net axial force transmitted by the core. The constant G represents the invariant flux of angular momentum transmitted by the core, which is equal to the torque transmitted from the source to the fluid.

An order-of-magnitude analysis of Eqs. (1-7) yields the following similarity relations for the case of strong swirl in the asymptotic region with $|w-W_0|\ll W_0$:

$$r=\left(\frac{4\nu z}{W_0}\right)^{1/2}\eta^{1/2},$$

$$u=\frac{F}{\rho(\nu W_0)^{1/2}z^{3/2}}i(\eta), \qquad v=\frac{G}{\rho W_0}\left(\frac{W_0}{\nu z}\right)^{3/2}h(\eta)$$

$$\frac{p_\infty-p}{\rho}=\left(\frac{G}{\rho W_0}\right)^2\left(\frac{W_0}{\nu z}\right)^3 k(\eta), \qquad w-W_0=\frac{F}{2\rho\nu z}f(\eta,S) \tag{8}$$

where η is the similarity coordinate and i, h, k, and f are nondimensional forms of the appropriate flow variables. The form f is a function also of the nondimensional parameter $S=G^2/\rho F(\nu z)^2$, where S represents the ratio of pressure gradient and inertial terms, and is a measure of the dynamical significance of the swirl-induced pressure deficit in the axial flow balance.

The reduced forms of Eqs. (1-4), in terms of the nondimensional variables, are

$$2\eta i'+i=\left(\eta f'+f+2S\frac{\partial f}{\partial S}\right)\eta^{1/2} \tag{9}$$

$$nh''+(\eta+1)h'+\left(\frac{3}{2}-\frac{1}{4\eta}\right)h=0 \tag{10}$$

$$nf''+(\eta+1)f'+f+2S\frac{\partial f}{\partial S}=2S(3k+\eta k') \tag{11}$$

$$2\eta k'=-h^2 \tag{12}$$

where primes denote differentiation with respect to η. The transformed boundary conditions are:

$$i=h=\eta^{1/2}f'=\eta^{1/2}k'=0, \quad f,k \text{ finite at } \eta=0$$

$$i\rightarrow 0, \quad h\rightarrow 0, \quad f\rightarrow 0, \quad k\rightarrow 0 \text{ as } \eta\rightarrow\infty \tag{13}$$

and the integral conditions (6) and (7) reduce to

$$2\pi\int_0^\infty (f+2S\eta k')\mathrm{d}\eta=1; \quad 8\pi\int_0^\infty \eta^{1/2}h\mathrm{d}\eta=1 \tag{14}$$

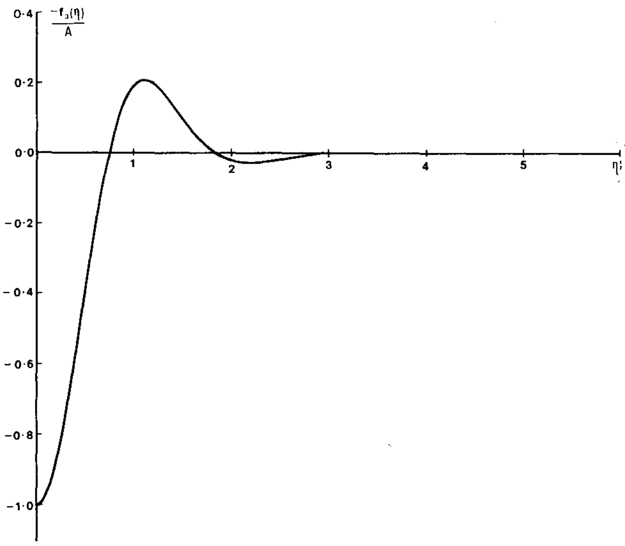

Fig. 1 The radial profile of the modification to the axial disturbance velocity, due to swirl-induced pressure coupling from the $S\ell n(ZW_0/\nu)$ term, in swirling jets and wakes. The curve is a nondimensional plot of $-f_3/A=1024\pi 2f_3/5=-e^{-\eta}(1-2\eta+\eta^2/2)$ against $\eta^{1/2}$.

Equation (10) may be integrated directly to give the general solution

$$h=D_1\eta^{1/2}e^{-\eta}+D_2\eta^{1/2}e^{-\eta}\int\frac{e^\eta}{\eta^2}\mathrm{d}\eta \tag{15}$$

where D_1 and D_2 are constants of integration. The second term in the solution is singular at $\eta=0$ and D_2 must be taken to be zero, and D_1 is obtained from the appropriate invariant integral (14) as $D_1=1/8\pi$.

Substitution of solution (15) into Eq. (12) yields, with the use of appropriate boundary conditions (13), the solution

$$k=\frac{1}{256\pi^2}e^{-2\eta}$$

Hence,

$$v=\frac{G}{8\pi\rho}\frac{W_0}{2\nu^2}\frac{r}{z^2}\exp\left[\frac{-r^2W_0}{4\nu z}\right]$$

$$\frac{p_\infty-p}{\rho}=\frac{G^2}{\rho}\frac{W_0}{(\nu z)^3}\frac{1}{256\pi^2}\exp\left[\frac{-r^2W_0}{2\nu z}\right] \tag{16}$$

represent solutions applicable for all ranges of swirl.

Assuming a solution to Eq. (11) in the form

$$f(\eta,S)=f_1(\eta)+Sf_2(\eta) \tag{17}$$

yields the linear equations

$$nf_1''+(\eta+1)f_1'+f_1=0 \tag{18}$$

$$nf_2''+(\eta+1)f_2'+3f_2=2(3k+\eta k') \tag{19}$$

subject to conditions derived from Eqs. (13) and (14):

$$\eta^{1/2}f_1'=\eta^{1/2}f_2'=0, \quad f_1,f_2 \text{ finite at } \eta=0$$

$$f_1\rightarrow 0, \quad f_2\rightarrow 0 \text{ as } n\rightarrow\infty$$

$$2\pi\int_0^\infty f_1\mathrm{d}\eta=1, \quad 2\pi\int_0^\infty (f_2+2\eta k')\mathrm{d}\eta=0 \tag{20}$$

The weak swirl solution, f_1, has been obtained by Newman[3] as

$$f_1 = \frac{1}{2\pi} e^{-\eta} \text{ or } w - W_0 = \frac{F}{2\pi\rho\nu z} \exp\left[\frac{-r^2 W_0}{4\nu z}\right] \quad (21)$$

The method of variation of parameters, plus some further reduction, yields the general solution of Eq. (19) as

$$f_2 = C_1 x_1 + C_2 x_2 + \frac{x_1}{256\pi^2} \int_a^\eta \frac{e^{-t}\left(\frac{77}{8} - \frac{65}{4}t - \frac{77}{4}t^2 + \frac{7}{2}t^3 - t^4\right)}{t[1-2t+(t^2/2)]} \mathrm{d}t \quad (22)$$

where C_1, C_2 are constants of integration, $a \neq 0$ is an arbitrary constant, and

$$x_1(\eta) = e^{-\eta}\left(1 - 2\eta + \frac{\eta^2}{2}\right); \quad x_2(\eta) = x_1 \int \frac{e^\eta \mathrm{d}\eta}{\eta(1-2\eta+\eta^2/2)^2} \quad (23)$$

Using x_2 is the form given in Eq. (23) it may be readily shown that f_2 (and hence w) is bounded at $\eta = 0$ only if

$$C_2 = \frac{-77}{8} \frac{1}{256\pi^2}$$

It is easily shown that the resulting solution (22) satisfies all boundary conditions (20) (including the invariant integral), with the arbitrary constant C_1 remaining undetermined.

The occurrence of an arbitrary constant in the solution for $f_2(\eta)$ is not surprising in itself. In fact, since the initial conditions have been ignored, we must anticipate indeterminacy at some stage of the expansion. Of greater concern is the fact that the second term of the right-hand side, whose coefficient is C_2, approaches zero as η^{-2} as $\eta \to \infty$, and not exponentially as we should expect to insure exponential vorticity decay in the freestream.

Similar problems have been found in a number of viscous flow problems involving asymptotic solutions. In the present case we will follow the manner used by the Berger[4] and assume a modified form of expansion (17) as

$$f(\eta, S) = f_1(\eta) + S[f_2(\eta) + \ln(zW_0/\nu) f_3(\eta)] \quad (24)$$

where the form of $f_3(\eta)$ is uniquely determined by requiring the resulting particular integral to be exponentially small for large η, although, as in Berger's solution, we shall see that the particular solution so defined is not unique.

Equation (11) now yields

$$\eta f_3'' + (\eta + 1) f_3' + 3f_3 = 0 \quad (25)$$

$$\eta f_2'' + (\eta + 1) f_2' + 3f_2 = f_3 + 2(3k + \eta k') \quad (26)$$

where $\eta^{1/2} f_2'(0) = \eta^{1/2} f_3'(0) = 0$, $f_2, f_3 \to 0$ exponentially as $\eta \to \infty$. It may be noted that the solution f_1 [Eq. (21)] remains unaltered by the modification.

As we have seen previously, the solution of Eq. (25) satisfying the boundary conditions is

$$f_3 = A[1 - 2\eta + (\eta^2/2)]e^{-\eta}$$

where A is an arbitrary constant. This result is now substituted into Eq. (26); the resulting equation can be readily solved by setting

$$f_2 = e^{-\eta}[1 - 2\eta + (\eta^2/2)]H(\eta) \quad (27)$$

$H(\eta)$ then satisfies the equation

$$H'' + \frac{\left(1 - 7\eta + \frac{9}{2}\eta^2 - \frac{\eta^3}{2}\right)}{\eta\left(1 - 2\eta + \frac{\eta^2}{2}\right)} H' = \frac{A}{\eta} + \frac{1}{128\pi^2} \frac{(3-2\eta)e^{-\eta}}{\eta\left(1 - 2\eta + \frac{\eta^2}{2}\right)}$$

A first integral of this equation is

$$H' e^{-\eta}\left(1 - 2\eta + \frac{\eta^2}{2}\right)^2 = \int_0^\eta e^{-t}\left(1 - 2t + \frac{t^2}{2}\right) \times \left[A\left(1 - 2t + \frac{t^2}{2}\right) + \frac{1}{128\pi^2}(3-2t)e^{-t}\right]\mathrm{d}t \quad (28)$$

If $f_2(\eta)$ is to tend to zero exponentially as $\eta \to \infty$, we must then require that

$$\int_0^\infty e^{-t}\left(1 - 2t + \frac{t^2}{2}\right)\left[A\left(1 - 2t + \frac{t^2}{2}\right) + \frac{1}{128\pi^2}(3-2t)e^{-t}\right]\mathrm{d}t = 0$$

This condition determines $A = -(5/1024\pi^2)$.

A further integration of Eq. (28) will yield a solution of the form

$$H = Q(\eta) + B$$

where B is an arbitrary constant. There seems little point in determining $Q(\eta)$, however, since substitution in Eq. (27) yields

$$f_2 = e^{-\eta}[1 - 2\eta + (\eta^2/2)]Q(\eta) + Be^{-\eta}[1 - 2\eta + (\eta^2/2)],$$

which satisfies all the boundary conditions with B remaining undetermined. Thus, f_3 is determined uniquely but f_2 is not; this is not surprising, since, as commented by Berger for his case, f_2 probably depends on the initial conditions. Indeed, in the present approach the only details of past events are supplied by invoking the invariant integral properties of the solution. In principle we could seek to overcome the indeterminacy in the solution by matching it with a near-wake solution. However, the lack of such a solution makes this impossible.

The expansion for the axial disturbance velocity yields

$$f = \frac{1}{2\pi} e^{-\eta} - S\ln\left(\frac{zW_0}{\nu}\right)\frac{5}{2048\pi^2} e^{-\eta}\left(1 - 2\eta + \frac{\eta^2}{2}\right) + Se^{-\eta}\left(1 - 2\eta + \frac{\eta^2}{2}\right)[Q(\eta) + B]$$

or

$$w = W_0 + \frac{F}{4\pi\rho\nu z}\exp\left[-\frac{r^2 W_0}{4\nu z}\right] - \frac{G^2}{2\rho^2(\nu z)^3}\exp\left[-\frac{r^2 W_0}{4\nu z}\right] \times \left[1 - 2\left(\frac{r^2 W_0}{4\nu z}\right) + \frac{1}{2}\left(\frac{r^2 W_0}{4\nu z}\right)^2\right]\left\{\frac{5}{2048\pi^2}\ln\left(\frac{zW_0}{\nu}\right) + Q(\eta) + B\right\} \quad (29)$$

Figure 1 shows the profile shape for the $S\ln(zW_0/\nu)$ term. For distances downstream large enough for the S term to be neglected relative to the $S\ln(zW_0/\nu)$ term, this profile shape represents approximately: the complete axial disturbance velocity profile for very strong swirling jets delivering zero flow force ($F \equiv 0$); and for $F \neq 0$ the modification to the Gaussian nonswirling jet or wake profile due to swirl-induced pressure coupling. In the latter case we observe that strong swirl acts to reduce axial velocity on and near the core axis. This effect is reversed as radial distance from the axis increases, with a further, but only slight, reversal occurring toward the outer region of the core. It is of interest to investigate whether this modification will be sufficient to shift the axial velocity maximum off the core axis for a large enough degree of swirl in the jet case. This requires $\partial^2 w/\partial r^2|_{r=0}>0$, $w|_{r=0}>W_0$ which, from Eq. (29) may be found to imply $S\ell n(W_0 z/\nu)>512\pi/15$, $S\ell n(W_0 z/\nu)<512\pi/5$, respectively. However, since $S_{max}\simeq 1$ and $S\sim z^{-2}$ it is difficult to see how the former condition can ever be met. On the other hand, the solution indicates that the far field of the momentumless jet ($F\equiv 0$) exhibits an axial velocity profile with maximum velocity off the centerline.

Acknowledgment

The author would like to thank Prof. B. R. Morton for his help and advice regarding this work.

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A Local Similarity Solution for the Viscous Boundary-Layer Flow Longitudinal to a Cylinder

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Nomenclature

a = radius of the cylinder
c = $a(w_0/\nu z)^{1/2}$ = curvature parameter
C_f = $2\tau_w/\rho w_0^2$ = dimensionless wall shear stress
f = w/w_0 = dimensionless axial velocity
r,z = radial and axial coordinates, respectively
Re_z = $w_0 z/\nu$ = Reynolds number
s = $\nu z/w_0 a^2$ = dimensionless axial coordinate
u,w = radial and axial velocity components, respectively
y = $r-a$ = distance above the cylinder surface
η = $y(w_0/\nu z)^{1/2}$ = Blasius similarity variable
ν = kinematic viscosity
ρ = density
τ = shear stress

Subscripts

0 = reference condition
w = conditions at the cylinder surface

Introduction

THE boundary-layer flow longitudinal to a cylinder is complicated by the transverse curvature of the surface, which precludes the possibility of obtaining a similarity solution to the governing boundary-layer equations. Since a similarity solution cannot be found, one naturally wonders whether a transformation of the boundary-layer equations exists that will minimize the streamwise variation of the velocity profiles. If such a transformation exists, the boundary-layer equations may, with some error resulting, be solved to obtain the local velocity profile and local surface shear stress without the necessity of accounting for the streamwise history of the boundary-layer flow.

The most accurate analysis of the problem of a uniform flow longitudinal to a semi-infinite, constant-radius cylinder appears to be that of Sparrow et al.,[1] in which a coordinate transformation was employed that was first suggested by Seban and Bond.[2] This coordinate transformation results in a dimensionless boundary-layer thickness that varies by a factor of 15 over the solution domain which, therefore, precludes the possibility of obtaining an accurate local similarity solution to the transformed equations. Presented in this Note is a coordinate transformation of the boundary-layer equations that substantially reduces the streamwise variation of the velocity profiles and the dimensionless boundary-layer thickness. The transformed equations can then be solved as ordinary differential equations to obtain the local velocity profile and local surface shear stress with an error or 2-5% resulting in the surface shear stress.

Analysis

The governing incompressible boundary-layer equations, written in cylindrical coordinates, are

$$\frac{\partial}{\partial r}(ru)+\frac{\partial}{\partial z}(rw)=0 \tag{1}$$

and

$$u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial r^2}\right) \tag{2}$$

The first term on the right-hand side of Eq. (2) introduces the transverse curvature effect that precludes the possibility of obtaining a similarity solution for this boundary-layer flow. Since a similarity solution does not exist, a dimensionless axial velocity component is defined as

$$f(\eta,s)=w/w_0$$

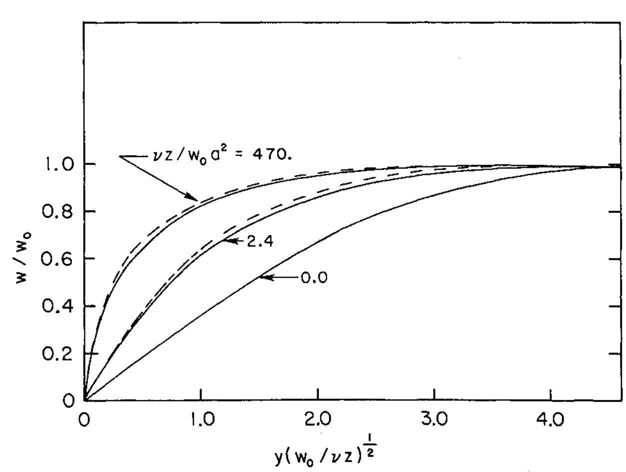

Fig. 1 Velocity profiles at various axial distances for the flow longitudinal to a constant-radius cylinder: (——) difference-differential solutions, (-----) local similarity solution.

Received April 9, 1983; revision received Aug. 8, 1983.

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